MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Bhaskaracharya Pratishthana, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 07/01/2024 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Find constants a, b such that $\lim_{x\to 0} \dfrac{\sqrt{ax+b}-2}{x} = 1$.
(A) $a=-4, b=-4$ (B) $a=4, b=4$ (C) $a=-3, b=4$ (D) $a=4, b=-2$.

2. The value of n for which $i+2(i)^2+3(i)^3+4(i)^4+\cdots+n(i)^n$ equals $-16+15i$ is
(A) 20 (B) 25 (C) 30 (D) 15.

3. Find the value of a real number b for which the sum of the squares of the zeros of $x^2-(b-2)x-b-1$ is minimal.
(A) 2 (B) 3 (C) -5 (D) 1.

4. Let S be the set of all last two digits of the powers of 3. (For example, $03, 09, 27, 81, 43 \in S$) Then, the number of distinct elements of S is
(A) 20 (B) 25 (C) 30 (D) 40.

5. The sum of the roots of $x^2-31x+220 = 2^x(31-2x-2^x)$ is
(A) 10 (B) 7 (C) 3 (D) 4.

6. The number of regions in which the plane gets divided by curves $(\sin t, \sin 2t)$ and $(\cos t, \cos 2t)$ for $t \in \mathbb{R}$ is
(A) 5 (B) 7 (C) 6 (D) 4.

7. In how many ways 6 persons can exchange seats among them in a row such that no one occupies his seat in a original position and exactly two of them have mutual exchange?
(A) $\binom{6}{2}\times 6$ (B) $\binom{6}{2}\times 4!$ (C) $\binom{6}{2}\times 3$ (D) $\binom{6}{2}\times 4$.

8. Let $A = \{x \in (0,3) \ : \ [x]^2 = [x^2]\}$, where $[x]$ denotes the greatest integer less than or equal to x. Let $M = \sup A$. Then
(A) $M \in A, M \notin (0,3)$ (B) $M \notin A, M \notin (0,3)$
(C) $M \in A, M \in (0,3)$ (D) $M \in A, M \in (0,3)$.

9. If $4047 = x + \dfrac{x}{1+2} + \dfrac{x}{1+2+3} + \dfrac{x}{1+2+3+4} + \cdots + \dfrac{x}{1+2+3+\cdots+4047}$, then $x =$
(A) 2000 (B) 2024 (C) 2002 (D) 2004.

10. Let $A(0,0), B(0,23), C(23,0)$ be the points in the plane. The number of points with integral coordinates that lie inside the triangle ABC (not on the boundary) is
(A) 253 (B) 242 (C) 231 (D) 219.

Part II

N.B. Each question in Part II carries 6 marks.

1. Following operations are permitted with a quadratic $ax^2 + bx + c$

   (i) Switch a and c,

   (ii) Replace x by $x + t$ for $t \in \mathbb{R}$.

   Is it possible to convert $x^2 - x - 2$ into $x^2 - x - 1$ by using the operations permitted? Justify.

2. Consider a right angled triangle PRQ with coordinates of the vertices integers. If slope and length of the hypotenuse PQ are integers, then show that PQ is parallel to the X-axis.

3. Let $1 \leq a_1 < a_2 < \cdots < a_\ell \leq n$ be integers with $\ell > \frac{n+1}{2}$. Show that there exist i, j, k with $1 \leq i < j < k \leq \ell$ such that $a_i + a_j = a_k$.

4. The sequence (a_n) is defined by

   $$a_1 = 0, |a_n| = |a_{n-1} + 1| \text{ for each } n \geq 2.$$

   For every positive integer n, prove that, $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq -\dfrac{1}{2}$.
   When does the equality hold?

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x(x-1)(2-x)$.

   $$\text{Let } S = \{x \in \mathbb{R} : f(x+t) > f(x) \text{ for some } t > 0\}.$$

   (a) Draw the graph of f.

   (b) Is the set S non-empty? Justify.

   (c) Is the set S bounded? Find $\sup S$, if it exists.

Part III

1. Let $\mathbb{Z}_{10}$ denote the set of integers modulo 10.

   (a) i. Find a nonzero solution to the following system of equations in $\mathbb{Z}_{10}$: [2]

   $$4x + 6y = 0$$
   $$2x + 4y = 0$$

   ii. Find a nonzero solution to the following system of equations in $\mathbb{Z}_{10}$: [2]

   $$4x + 3y = 0$$
   $$x + 2y = 0$$

   (b) Prove that the system of equations

   $$ax + by = 0$$
   $$cx + dy = 0$$

   has a unique solution $x = 0, y = 0$ in $\mathbb{Z}_{10}$ if and only if the number $(ad - bc) \pmod{10} \in \{1, 3, 7, 9\}$. [8]

2. Let $f(x) = a_0 + a_1x + a_2x^2 + a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + a_{13}x^{13}$ and $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_{11}x^{11} + b_{12}x^{12} + b_{13}x^{13}$ be polynomials with real coefficients such that $a_{13} \neq 0, b_3 \neq 0$. Prove that the degree of $gcd(f, g) \leq 6$. [12]

3. (a) Let x_0 be an arbitrary real number. Define a sequence (x_n) as follows:

$$x_n = \frac{x_{n-1} + 4}{5}, \forall n \geq 1.$$

Show that sequence (x_n) is convergent. Find $\lim_{n\to\infty} x_n$. [5]

(b) Let n be a fixed positive integer. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-zero function satisfying following conditions:

i n^{th} derivative of f is continuous.

ii $f(5x + 3) = 5^n f\left(x + \frac{7}{5}\right), \forall x \in \mathbb{R}$.

Show that f is a polynomial of degree n. [8]

4. Let n be a positive integer greater than 1. Let $\rho(n)$ be the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

(a) Find $\rho(2)$ and $\rho(3)$. [4]

(b) Find $\rho(4)$. [3]

(c) Find $\rho(n)$ for each n. [6]