MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S.P.College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 13/12/2015 Max. Marks:100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Let $A(t)$ denote the area bounded by the curve $y = e^{-|x|}$, the $X-$ axis and the straight lines $x = -t, x = t$, then $\lim_{t\to\infty} A(t)$ is
A) 2 B) 1 C) 1/2 D) e.

2. How many triples of real numbers (x, y, z) are common solutions of the equations
$x + y = 2, \quad xy - z^2 = 1$?
A) 0 B) 1 C) 2 D) infinitely many.

3. For non-negative integers x, y the function $f(x, y)$ satisfies the relations $f(x, 0) = x$ and $f(x, y + 1) = f(f(x, y), y)$. Then which of the following is the largest?
A) $f(10, 15)$ B) $f(12, 13)$ C) $f(13, 12)$ D) $f(14, 11)$.

4. Suppose p, q, r, s are $1, 2, 3, 4$ in some order. Let $x = \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s}}}}$
We choose p, q, r, s so that x is as large as possible, then s is
A) 1 B) 2 C) 3 D) 4.

5. Let $f(x) = \begin{cases} 3x + x^2 & \text{if } x < 0 \\ x^3 + x^2 & \text{if } x \geq 0. \end{cases}$ Then $f''(0)$ is
A) 0 B) 2 C) 3 D) None of these.

6. There are 8 teams in pro-kabaddi league. Each team plays against every other exactly once. Suppose every game results in a win, that is, there is no draw. Let $w_1, w_2, \cdots, w_8$ be number of wins and $l_1, l_2, \cdots, l_8$ be number of loses by teams $T_1, T_2, \cdots, T_8$, then
A) $w_1^2 + \cdots + w_8^2 = 49 + (l_1^2 + \cdots + l_8^2)$. B) $w_1^2 + \cdots + w_8^2 = l_1^2 + \cdots + l_8^2$.
C) $w_1^2 + \cdots + w_8^2 = 49 - (l_1^2 + \cdots + l_8^2)$. D) None of these.

7. The remainder when $m + n$ is divided by 12 is 8, and the remainder when $m - n$ is divided by 12 is 6. If $m > n$, then the remainder when mn divided by 6 is
A) 1 B) 2 C) 3 D) 4 .

8. Let $A = \begin{pmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \ddots & & \vdots \\ (n-1)n+1 & (n-1)n+2 & \dots & n^2 \end{pmatrix}$. Select any entry and call it x_1.
Delete row and column containing x_1 to get an $(n-1) \times (n-1)$ matrix. Then select any entry from the remaining entries and call it x_2. Delete row and column containing x_2 to get $(n-2) \times (n-2)$ matrix. Perform n such steps. Then $x_1 + x_2 + \cdots + x_n$ is
A) n B) $\dfrac{n(n+1)}{2}$ C) $\dfrac{n(n^2+1)}{2}$ D) None of these.

9. The maximum of the areas of the rectangles inscribed in the region bounded by the curve $y = 3 - x^2$ and $X-$axis is
A) 4 B) 1 C) 3 D) 2.

10. How many factors of $2^5 3^6 5^2$ are perfect squares?
A) 24 B) 20 C) 30 D) 36.

Part II

N.B. Each question in Part II carries 6 marks.

1. How many 15−digit palindromes are there in each of which the product of the non-zero digits is 36 and the sum of the digits is equal to 15? (A string of digits is called a palindrome if it reads the same forwards and backwards. For example 04340, 6411146.)

2. Let H be a finite set of distinct positive integers none of which has a prime factor greater than 3. Show that the sum of the reciprocals of the elements of H is smaller than 3. Find two different such sets with sum of the reciprocals equal to 2.5.

3. Let A be an $n \times n$ matrix with real entries such that each row sum is equal to one. Find the sum of all entries of A^{2015}.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f'(x) > f(x)$ for all $x \in \mathbb{R}$. Prove that $f(x) > 0$ for all $x > 0$.

5. Give an example of a function which is continuous on $[0, 1]$, differentiable on $(0, 1)$ and not differentiable at the end points. Justify.

Part III

1. There are some marbles in a bowl. A, B and C take turns removing one or two marbles from the bowl, with A going first, then B, then C, then A again and so on. The player who takes the last marble from the bowl is the loser and the other two players are the winners. If the game starts with N marbles in the bowl, for what values of N can B and C work together and force A to lose? [12]

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f'(0)$ exists. Suppose $\alpha_n \neq \beta_n, \forall n \in \mathbb{N}$ and both sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge to zero. Define $D_n = \dfrac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$.
Prove that $\lim_{n\to\infty} D_n = f'(0)$ under EACH of the following conditions:

   (a) $\alpha_n < 0 < \beta_n, \ \forall n \in \mathbb{N}$.

   (b) $0 < \alpha_n < \beta_n$ and $\dfrac{\beta_n}{\beta_n - \alpha_n} \leq M, \quad \forall n \in \mathbb{N}$, for some $M > 0$.

   (c) $f'(x)$ exists and is continuous for all $x \in (-1, 1)$.

   [13]

3. Let $f(x) = x^5$. For $x_1 > 0$, let $P_1 = (x_1, f(x_1))$. Draw a tangent at the point P_1 and let it meet the graph again at point P_2. Then draw a tangent at P_2 and so on. Show that the ratio $\dfrac{A(\triangle P_n P_{n+1} P_{n+2})}{A(\triangle P_{n+1} P_{n+2} P_{n+3})}$ is constant. [12]

4. Let $p(x)$ be a polynomial with positive integer coefficients. You can ask the question: What is $p(n)$ for any positive integer n? What is the minimum number of questions to be asked to determine $p(x)$ completely? Justify. [13]