Part I

N.B. Each question in Part I carries 2 marks.

1. If $p(x)$ is a non-constant polynomial, then $\lim\limits_{k\to\infty} \dfrac{p(k+1)}{p(k)}$ is equal to
   (a) 1 (b) 0 (c) -1 (d) the leading coefficient of $p(x)$.

2. The number of continuous functions f from $[-1,1]$ to $\mathbb{R}$ satisfying $(f(x))^2 = x^2$ for all $x \in [-1,1]$ is
   (a) 2 (b) 3 (c) 4 (d) infinite.

3. Let $q \in \mathbb{N}$. The number of elements in set $\{(\cos\frac{\pi}{q} + i\sin\frac{\pi}{q})^n \mid n \in \mathbb{N}\}$ is
   (a) 1 (b) q (c) infinite (d) $2q$.

4. If $f(x) = |x|^{\frac{3}{2}}$, $\forall x \in \mathbb{R}$, then at $x = 0$,
   (a) f is not continuous (b) f is continuous but not differentiable (c) f is differentiable but f' is not continuous (d) f is differentiable and f' is continuous.

5. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are roots of the equation $x^4 + x^3 + 1 = 0$, then the value of $(1-2\alpha_1)(1-2\alpha_2)(1-2\alpha_3)(1-2\alpha_4)$ is equal to
   (a) 19 (b) 16 (c) 15 (d) 20.

6. If A and B are 3×3 real matrices with rank $(AB) = 1$, then rank (BA) cannot be
   (a) 0 (b) 1 (c) 2 (d) 3.

7. The number of common solutions of $x^{36} - 1 = 0$ and $x^{24} - 1 = 0$ in the set of complex numbers is
   (a) 1 (b) 2 (c) 6 (d) 12.

8. If f is a one to one function from $[0,1]$ to $[0,1]$, then
   (a) f must be onto (b) f cannot be onto (c) $f([0,1])$ must contain a rational number (d) $f([0,1])$ must contain an irrational number.

9. There are 18 ways in which n identical balls can be grouped such that each group contains equal number of balls. Then the minimum value of n is
   (a) 120 (b) 180 (c) 160 (d) 90.

10. Suppose f and g are two linear functions as shown in the figure. Then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$
    (a) is 2 (b) does not exist (c) is 3 (d) is $\frac{1}{2}$.

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Part II

N.B. Each question in Part II carries 5 marks. Attempt any FOUR:

(a) Let $A = (a_{ij})$ be $n \times n$ matrix, where $a_{ij} = \max\{i, j\}$. Find the determinant of A.

(b) Assume that f is a continuous function from $[0, 2]$ to $\mathbb{R}$ and $f(0) = f(2)$. Prove that there exist x_1 and x_2 in $[0, 2]$ such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$.

(c) Let $X = \{1, 2, 3, \cdots, 10\}$. Determine the number of ways of expressing X as $X = A_1 \cup A_2 \cup A_3$, where $A_1, A_2, A_3 \subseteq X$ and $A_1 \cap A_2 \cap A_3 = \phi$.

(d) For non-negative real numbers $a_1, a_2, \cdots, a_n$, show that $\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} a_k e^{-a_k} \leq \dfrac{1}{e}$.

(e) Let A be a non-zero $1 \times n$ real matrix. Then show that the rank of $A^t A$ is 1.
[Note that A^t denotes the transpose of the matrix A.]

Part III

N.B. Each question in Part III carries 12 marks. Attempt any FIVE:

(a) Let $f(x) = \dfrac{(x-a)(x-b)}{(x-c)}, \quad x \neq c.$
Find the range of f in each of the following cases:
i] $a < c < b$ ii] $c < a < b$.

(b) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.

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(c) Suppose $A_1, A_2, \cdots, A_n$ are vertices of a regular n-gon inscribed in a unit circle and P is any point on the unit circle. Prove that $\displaystyle\sum_{i=1}^{n} l(PA_i)^2$ is constant, where $l(PA_i)$ denotes the distance between P and A_i. [Hint: Use complex numbers.]

(d) A unit square of a chess board of size $n \times n$ gets infected if at least two of its neighbours are infected. Find the maximum number of infected unit squares if initially [i] 2 unit squares are infected, [ii] 3 unit squares are infected. Find the minimum number of unit squares that should be infected initially so that the whole chess board gets infected.
(Two unit squares are called neighbours if they share a common edge.)

(e) Let a, b, c be integers. Let $x = \dfrac{p}{q}, y = \dfrac{r}{s}$ be rational numbers satisfying $y^2 = x^3 + ax^2 + bx + c$. Show that there exists an integer t such that $q = t^2, s = t^3$.

(f) Show that the polynomial $p_n(x) = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}$ has no real root if n is even and exactly one real root if n is odd.