Entrance Examination for BSc Programmes at CMI, May 2012

Attempt all 5 problems in part A, each worth 6 points. Attempt 7 out of the 9 problems in part B, each worth 10 points.

Part A. (5 problems $\times$ 6 points = 30 points.) Clearly explain your entire reasoning.

1. Find the number of real solutions to the equation $x = 99\sin(\pi x)$.

2. A differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(1) = 2, f(2) = 3$ and $f(3) = 1$. Show that $f'(x) = 0$ for some x.

3. Show that $\frac{\ln(12)}{\ln(18)}$ is irrational.

4. Show that

$$\lim_{x\to\infty} \frac{x^{100}\ln(x)}{e^x \tan^{-1}(\frac{\pi}{3} + \sin x)} = 0\,.$$

5. (a) n identical chocolates are to be distributed among the k students in Tinku's class. Find the probability that Tinku gets at least one chocolate, assuming that the n chocolates are handed out one by one in n independent steps. At each step, one chocolate is given to a randomly chosen student, with each student having equal chance to receive it.

(b) Solve the same problem assuming instead that all distributions are equally likely. You are given that the number of such distributions is $\binom{n+k-1}{k-1}$. (Here all chocolates are considered interchangeable but students are considered different.)

Part B. (9 problems $\times$ 10 points = 90 points.) Clearly explain your entire reasoning.

Attempt at least 7 problems. You may solve only part of a problem and get partial credit. If you cannot solve an earlier part, you may assume it and proceed to the next part. For all such partial answers, clearly mention what you are solving and what you are assuming.

1. a) Find a polynomial $p(x)$ with *real* coefficients such that $p(\sqrt{2} + i) = 0$.

b) Find a polynomial $q(x)$ with *rational* coefficients and having the smallest possible degree such that $q(\sqrt{2} + i) = 0$. Show that any other polynomial with rational coefficients and having $\sqrt{2} + i$ as a root has $q(x)$ as a factor.

2. a) Let E, F, G and H respectively be the midpoints of the sides AB, BC, CD and DA of a convex quadrilateral ABCD. Show that EFGH is a parallelogram whose area is half that of ABCD.

b) Let E = $(0, 0)$, F = $(0, -1)$, G = $(1, -1)$, H = $(1, 0)$. Find all points A = (p, q) in the first quadrant such that E, F, G and H respectively are the midpoints of the sides AB, BC, CD and DA of a convex quadrilateral ABCD.

3. a) We want to choose subsets $A_1, A_2, \ldots, A_k$ of $\{1, 2, \ldots, n\}$ such that any two of the chosen subsets have nonempty intersection. Show that the size k of any such collection of subsets is at most 2^{n-1}.

b) For $n > 2$ show that we can always find a collection of 2^{n-1} subsets $A_1, A_2, \ldots$ of $\{1, 2, \ldots, n\}$ such that any two of the A_i intersect, but the intersection of all A_i is empty.

4. Define

$$x = \sum_{i=1}^{10} \frac{1}{10\sqrt{3}} \frac{1}{1+(\frac{i}{10\sqrt{3}})^2} \qquad \text{and} \qquad y = \sum_{i=0}^{9} \frac{1}{10\sqrt{3}} \frac{1}{1+(\frac{i}{10\sqrt{3}})^2}.$$

Show that a) $x < \frac{\pi}{6} < y$ and b) $\frac{x+y}{2} < \frac{\pi}{6}$. (Hint: Relate these sums to an integral.)

5. Using the steps below, find the value of $x^{2012} + x^{-2012}$, where $x + x^{-1} = \frac{\sqrt{5}+1}{2}$.

a) For any *real* r, show that $|r + r^{-1}| \geq 2$. What does this tell you about the given x?

b) Show that $\cos(\frac{\pi}{5}) = \frac{\sqrt{5}+1}{4}$, e.g. compare $\sin(\frac{2\pi}{5})$ and $\sin(\frac{3\pi}{5})$.

c) Combine conclusions of parts a and b to express x and therefore the desired quantity in a suitable form.

6. For $n > 1$, a *configuration* consists of $2n$ distinct points in a plane, n of them red, the remaining n blue, with no three points collinear. A *pairing* consists of n line segments, each with one blue and one red endpoint, such that each of the given $2n$ points is an endpoint of exactly one segment. Prove the following.

a) For any configuration, there is a pairing in which no two of the n segments intersect. (Hint: consider total length of segments.)

b) Given n red points (no three collinear), we can place n blue points such that any pairing in the resulting configuration will have two segments that do not intersect. (Hint: First consider the case $n = 2$.)

7. A sequence of integers c_n starts with $c_0 = 0$ and satisfies $c_{n+2} = ac_{n+1} + bc_n$ for $n \geq 0$, where a and b are integers. For any positive integer k with $gcd(k, b) = 1$, show that c_n is divisible by k for infinitely many n.

8. Let $f(x)$ be a polynomial with integer coefficients such that for each nonnegative integer n, $f(n) =$ a perfect power of a prime number, i.e., of the form p^k, where p is prime and k a positive integer. (p and k can vary with n.) Show that f must be a constant polynomial using the following steps or otherwise.

a) If such a polynomial $f(x)$ exists, then there is a polynomial $g(x)$ with integer coefficients such that for each nonnegative integer n, $g(n) =$ a perfect power of a *fixed* prime number.

b) Show that a polynomial $g(x)$ as in part a must be constant.

9. Let N be the set of *non-negative* integers. Suppose $f : N \to N$ is a function such that $f(f(f(n))) < f(n+1)$ for every $n \in N$. Prove that $f(n) = n$ for all n using the following steps or otherwise.

a) If $f(n) = 0$, then $n = 0$.

b) If $f(x) < n$, then $x < n$. (Start by considering $n = 1$.)

c) $f(n) < f(n+1)$ and $n < f(n+1)$ for all n.

d) $f(n) = n$ for all n.