2015

BOOKLET NO. TEST CODE: **QRB**

Afternoon **Time: 2 hours**

Read the following carefully before answering the test.

Write your Registration Number, Test Code, Booklet No. etc., in the appropriate places on the answer-booklet.

The question paper is divided into **two** groups: **Group A** and **Group B**.

- Group A has **six** questions on **Mathematics** and **Probability & Statistics**.
- Group B has **nine** questions on **Operations Research, Reliability** and **Statistical Quality Control & Quality Management**.

Each question carries a total of 20 marks. You have to answer any **six questions**, taking **at least two questions from each group**. Marks allotted to each part question is given within [ ].

- **Partial credit may be given for partial answer.**
- **Full credit will be given for complete and rigorous arguments.**

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR ON THE ANSWER-BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATORS IN ANY FORM.

STOP! WAIT FOR THE SIGNAL TO START.

Group A

Mathematics

1. (a) Suppose for real numbers a, b, c and d,

$$\begin{aligned} a+b+c+d &= 0 \\ \text{and}\quad a^3+b^3+c^3+d^3 &= 0. \end{aligned}$$

Show that

$$a^{2m+1}+b^{2m+1}+c^{2m+1}+d^{2m+1}=0$$

for all natural numbers $m > 1$. [12]

(b) Let A be an $n \times n$ real symmetric matrix, and

$$A^{2m} \longrightarrow C \quad \text{as} \quad m \to \infty$$

with element-wise convergence, where C is some real $n \times n$ matrix. Show that only 0 or 1 can be an eigenvalue of C.

[8]

2. (a) Consider

$$f(x) \;=\; \exp(x^2/2)\int_x^\infty \exp(-u^2/2)du \qquad \text{for } x > 0.$$

(i) Prove that $f(x)$ satisfies: $0 < f(x) < 1/x$. [5]

(ii) Show that $f(x)$ is a strictly decreasing function of x.

[3]

(b) A small town has three schools (X, Y and Z) and a coaching center. All students of school X play cricket. Those students of the town who play cricket also play football, but do not play volleyball. All students of the coaching center play basketball. Those basketball players of the town who play football also play volleyball. Prove that no student of school X is a student of the coaching center. [5]

(c) Find the positive integer(s) which leave remainder 6 when divided by 22, and 12 when divided by 39. [7]

3. (a) Suppose for $0 < \theta, \phi < \pi/2$,

$$x = \sum_{n=0}^{\infty} \sin^{2n} \theta,$$

$$y = \sum_{n=0}^{\infty} \cos^{2n} \phi$$

and $$z = \sum_{n=0}^{\infty} \cos^{n}(\theta + \phi) \cos^{n}(\theta - \phi).$$

Show that $xyz - xy = yz - zx$. [10]

(b) Suppose that a function f satisfies the following conditions for all real values of x and y:

(i) $f(x+y) = f(x) \cdot f(y)$,

(ii) $f(x) = 1 + x \cdot g(x)$ where $\lim_{x \to 0} g(x) = 1$.

Show that $\ln f(x) = x$. [5]

(c) Evaluate the integral:

$$\int_{1/2}^{2} \frac{1}{x} \operatorname{cosec}^{101}\left(x - \frac{1}{x}\right) dx.$$

[5]

Probability & Statistics

4. (a) In a TV show, A & B agree to play as follows. They throw two die and if sum 'S' of the outcomes is less than 10 then B earns Rs. S from the sponsor, otherwise A earns fixed amount Rs. x. Find the value of x so that expected earning for each of them is the same. [5]

(b) In another TV show involving A & B, A gets a prize of random amount X following an exponential distribution with mean $1/\lambda$ and B gets a prize of fixed amount c. Find the value of c such that the absolute difference between the two prize amounts is minimum. [4]

(c) In a test, an examinee either guesses or copies or knows the answers to a multiple choice question which has four choices with one correct answer. The probability that he makes a random guess is $\frac{1}{3}$ and that he copies the answer is $\frac{1}{6}$. The probability that his answer is correct given that he has copied it is $\frac{1}{8}$. Find the probability that he knows the answer to the question given that he has answered it correctly. [4]

(d) The random variables (X, Y) have the joint probability density function

$$f(x,y) = C \cdot \exp\left[-\frac{8x^2}{3} + \frac{32x}{3} - \frac{32}{3} - 6y^2 + 4xy - 8y\right],$$

$-\infty < x, y < \infty$. Find the value of C and the correlation coefficient between X and Y. [7]

5. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $N(\mu, 1)$.

(a) Find the maximum likelihood estimator of μ^2. Derive its variance. [3+7=10]

(b) Find an unbiased estimator of μ^2. Compute the Cramer-Rao lower bound for the variance of this estimator.

[3+7=10]

6. (a) Consider the linear model:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i,$$

where $\epsilon_i \sim$ i.i.d. $N(0, \sigma^2)$, $i = 1, 2, \ldots, n$.

(i) Obtain a test for the hypothesis $\beta_1 = \beta_2$, explaining precisely the distribution of the test statistic under the null hypothesis and the cut-off to be used in order to ensure level $\alpha = 0.05$. [10]

(ii) Obtain the best linear unbiased estimator of β_1 under the restriction $\beta_1 = \beta_2$. [4]

(b) Consider a coin-tossing experiment with two coins, Coin 1 and Coin 2, having probability of Head as 0.7 and 0.6, respectively. If the coin tossed today produces Head, then Coin 1 is selected for tomorrow's toss and if the coin tossed today produces Tail then Coin 2 is selected for tomorrow's toss. Let X_n be the label (Coin 1 or Coin 2) of the coin tossed on the nth day. Present $\{X_n\}$ as a Markov chain by writing the corresponding transition probability matrix and hence obtain the equilibrium probability of the individual states. [3+3=6]

Group B

Operations Research

7. (a) A game is completely mixed if no optimal strategy of either player can skip a row or column. Let A be a real square pay-off matrix whose off-diagonal entries are non-positive. If the value of the game is positive, then show that the game is completely mixed. [5]

   (b) Find out whether the system $Ax \leq 0$ and $c^t x > 0$ has a solution $x \in R^3$, where

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}.$$

[7]

   (c) Let Ω be a bounded convex polyhedron region given by the extreme points (or corner points) $P_1(1,3)$, $P_2(4,0)$, $P_3(5,2)$ and $P_4(5,4)$. Let P_0 be the point (-2,-1). Formulate the problem of finding a point in Ω that is closest to P_0, in terms of usual Euclidean distance, as an optimization problem.

[8]

8. (a) Suppose that $S \subseteq R^n$ is a convex set, and $f : S \to R$. Let the epigraph of the function f be defined as *epi* $(f) = \{(x,y) : y \geq f(x), x \in S, y \in R\} \subseteq R^{n+1}$. Show that f is a convex function if *epi* (f) is a convex set. Is the converse true? Justify. [4+2=6]

   (b) Suppose that m data points of the form $(a_i, b_i), i = 1, \ldots, m$ are given, where $a_i \in R^n$ and $b_i \in R$. Let x be a vector of unknown parameters to be determined so that the largest error, $\max_{1 \le i \le m} |b_i - a_i^t x|$, is minimized. Show that this problem is equivalent to a linear programming problem. [7]

(c) Consider the following quadratic programming problem:

$$\begin{aligned} &\text{Minimize } f(x) = c^t x + \frac{1}{2} x^t Q x \\ &\text{subject to} \quad Ax \ge b, \; x \ge 0, \end{aligned}$$

where $Q \in R^{n \times n}$ is symmetric, $A \in R^{m \times n}$, $b \in R^m$ and $c \in R^n$. Also, suppose that f is a convex function. Write down the Karush-Kuhn-Tucker (KKT) conditions for the problem. Is the KKT point a global optimal solution for the problem? Justify. [5+2 = 7]

9. (a) Suppose that $A \in R^{m \times n}, b \in R^m$ and $c \in R^n$ are given. Consider the linear programming problem:

$$\begin{aligned} &\text{Minimize } f(x) = c^t x - b^t y \\ &\text{subject to} \quad Ax \qquad \ge \; b \\ &\qquad\qquad\quad - A^t y \ge -c \\ &\qquad\qquad\quad x \ge 0, \; y \ge 0, \end{aligned}$$

where the decision variables x and y are vectors of appropriate dimensions. Show that the above problem is either infeasible or has an optimal solution with objective value zero. [7]

(b) Solve an assignment problem with the cost matrix $C = ((c_{ij}))$, where $c_{ij} = r_i + s_j$ with r_i and s_j being known constants for $i, j = 1, 2, \ldots, n$. [7]

(c) Let $f_i : R^n \to R$ be convex function for $i = 1, 2, \ldots, m$. Show that the function f defined by $f(x) = \max_{1 \le i \le m} \{f_i(x)\}$ is also convex. [6]

Reliability

10. (a) Consider a coherent system with four components. The cut sets of the system are $\{1,2\}$, $\{3,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$ and $\{1,2,3,4\}$.

    (i) Find the min cut sets and min path sets. [1+2=3]

    (ii) Write down the structure function of its dual system using the min path sets of the dual. [3]

    (iii) Suppose the lifetime of the ith component has constant hazard rate λ_i, $i = 1, \ldots, 4$. Derive the reliability function of the system and its mean residual life at time t_0.

    [5+3=8]

    (b) A light bulb has a constant hazard rate λ_1 or λ_2 depending on if it is of Brand 1 or Brand 2, rspectively. Consider a lot consisting of bulbs of both the brands in equal numbers. Obtain hazard rate for the life of a bulb randomly chosen from the lot. Prove that the corresponding life distribution is DFR. [3+3=6]

11. (a) Consider a life testing experiment with n items in which one observes the number of failures d by a pre-specified time T_0. Suppose the lifetimes of the items are independent and have the probability density function

$$f(t) = 2\lambda t e^{-\lambda t^2}, \quad t > 0, \lambda > 0.$$

    (i) Find the distribution of d. [2]

    (ii) Find the maximum likelihood estimate of λ from the above observation. [3]

    (iii) Suggest a confidence interval for λ with at least 90% confidence level. [5]

(b) Suppose n identical devices are put on a life test at time $t = 0$. The life test is continued until a pre-fixed number r ($\leq n$) of the devices have failed (the remaining are censored at the r-th failure time). Suppose the lifetimes of the items follow independent and identical distribution with constant hazard rate λ. Find the expected duration of the experiment. [4]

(c) If the mean remaining life of a continuous life distribution is $r(t)$, derive an expression for the hazard rate of the distribution in terms of $r(t)$. [6]

12. (a) In a two-unit standby redundant system, the failure rate for the operating unit is 0.12 per day. The non-operating unit can fail while in standby with a failure rate 0.02 per day. When the operating unit fails, the standby unit, if intact, is pressed into operation through an automatic switch. Once operational, the second unit has a failure rate of 0.12 per day.

    (i) Obtain the reliability function for the system, assuming that the switch is perfect. [6]

    (ii) Obtain the reliability function for the system, assuming that the probability of failure of the switch is 0.05. [2]

    (iii) Find the mean time to failure of the system in part (ii). [4]

(b) (i) Consider a system with two components. The two components, while operating together, are independent having exponential life distribution with mean $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$, respectively. As soon as one of them fails, both the components are replaced instantaneously with independent and identical spares of the two components. Obtain the expected number of replacements by time t. [4]

(ii) Consider data on successive replacement times and the identity of the failed component (1 or 2) till n replacements take place. Describe the data formally and obtain the maximum likelihood estimates of λ_1 and λ_2.

[1+3=4]

Statistical Quality Control & Quality Management

13. (a) Suppose that $X_1, X_2, \ldots$ denote the measurements made on parts as they are produced successively. In order to devise a process monitoring system, consider the following statistics:

$$Q_r = \left(\frac{r-1}{r}\right)^{\frac{1}{2}} \frac{X_r - \bar{X}_{r-1}}{\sigma_0}, \quad r = 2, 3, \ldots$$

where X_i's are i.i.d. $N(\mu, \sigma_0^2)$ with known σ_0 and $\bar{X}_{r-1} = \sum_{i=1}^{r-1} X_i/(r-1)$. Find the distribution of Q_r's, and hence suggest a control chart for the production process based on Q_r's. [8]

(b) Consider a sample of 20 units drawn randomly from a stable production process. An inspector measures each unit twice. Let R_i be the range of readings on ith unit. R-chart exhibits in-control situation, where $\bar{R}$ (average of R_i's) is 1. Standard deviation of all 40 readings is computed as $S = 3.17$. Estimate the process capability after eliminating the effect of measurement error. [Given that $d_2 = 1.128, 3.735$ for $n = 2, 20$, respectively]. [6]

(c) Explain how one can estimate C_p and C_{pk} in presence of autocorrelation. [6]

14. A company recently increased its production capacity two times. The available inspection staff became insufficient to carry out the inspection of the increased volume of incoming materials. However, the top management of the company decided not to increase the inspection staff, and takes a policy decision that only a fixed sample size n shall be inspected from each lot at the receiving inspection irrespective of the lot sizes. Under this circumstances, as the Quality Assurance Manager you need to choose a best compromise sampling plan based upon the following criterion: the sum of the probabilities of accepting lots of true quality p_1 (acceptable quality) and rejecting lots of true quality p_2 (objectionable quality) be a maximum (assume binomial population).

   (i) Derive the functional relationships that must hold for c to be the acceptance number that maximizes the said criterion. [8]

   (ii) Show, using the relationships derived in (i), that optimal acceptance number c^* satisfies the following:

$$k-1 \leq c^* \leq k, \text{ for } k = \frac{n \log \frac{1-p_1}{1-p_2}}{\log \frac{p_2}{p_1} + \log \frac{1-p_1}{1-p_2}}.$$

[12]

15. (a) Successful organizations often fail to identify changes in customer preferences even when persons involved in direct sales can get a feel of the impending directional change. This phenomenon has been attributed to the failure of effective implementation of quality management principles. Identify the principle that has probably been violated. Give reasons for your answer. Clearly state the assumptions you are making about the structure of the organization. [8]

   (b) Public service organizations generally do not have a quality department for the design and monitoring of services, but there are often overseeing departments where complaints regarding quality of services may be made. It has been empirically observed that this system rarely delivers results. Identify two reasons why the system of complaining to a higher authority may fail to improve the quality of services.

[7]

(c) Attempting to increase senior management focus on quality through computation and reporting of cost of quality is a contentious issue. It has often been argued that this approach leads to defensive rather than strategic approach to quality. Do you agree? Explain.

[5]