2015

BOOKLET NO. TESTCODE : PHB

Afternoon

Questions : 6+8+8	**Time : 2 hours**

On the answer-booklet write your Registration Number, Test Code, Number of this booklet, etc. in the appropriate places.

ATTENTION !

Please read the following very carefully before answering the test.

There are three parts. Part I is *compulsory* for all candidates and carries a credit of 30 % of the total. Besides, each candidate has to choose *only one* of the Parts II & III and answer from that part as per instructions. The credit for each of these parts is 70 % of the total.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR ON THE ANSWER-BOOKLET. YOU ARE NOT ALLOWED TO USE CALCULATORS.

STOP ! WAIT FOR THE SIGNAL TO START

Part I

Mathematical and Logical Reasoning

Answer all questions. All questions carry equal marks.

1. A point P moves along the curve $xy = c$ with O as the origin. Find the co-ordinates of P so that OP is minimum.

2. Find the number of ways of distributing 4 distinct balls into 3 identical boxes so that no box is empty.

3. Let $A = (a_{ij})$ be a 17×17 matrix with entries a_{ij} defined as follows.
$$a_{ij} = \begin{cases} +1 & \text{if } i > j \\ 0 & \text{if } i = j \\ -1 & \text{if } i < j \end{cases}$$
Is A invertible? Justify your answer.

4. A spaceship is on a straight line path between Earth and Moon. At what distance from the Earth the net gravitational force on the spaceship is zero?

5. Find the most general solution for the following set of coupled differential equations
$$\frac{d^2x(t)}{dt^2} + Ax(t) - By(t) = 0 \;\text{ and }\; \frac{d^2y(t)}{dt^2} + Ay(t) - Bx(t) = 0$$
where A and B are two constants.

6. A block of unknown mass is attached to a spring of spring constant 6.5 N/m and undergoes simple harmonic motion with an amplitude of 10 cm. When the block is halfway between its equilibrium position and the end point, its speed is measured to be 30 cm/sec. Calculate the mass of the block.

Part II

Mathematics

Answer any five questions.

1. (a) Using a suitable contour show that

$$\int_0^\infty \frac{\cos(kx)}{x^2+a^2} = \frac{\pi}{2a}e^{-|k|a}, \text{ where } k,\ a > 0 \text{ are real numbers.}$$

(b) Consider a transformation $w = f(z)$ where $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$. Show that under this transformation the tangent at z_0 to any curve C in the z plane passing through z_0 is rotated through an angle $\arg[f'(z_0)]$.

[7+7]

2. (a) A projectile is fired from a platform that is moving horizontally with velocity V. The initial velocity of the projectile relative to the platform is $\mathbf{u} = (u_1, u_2, 0)$. Show that the range on a horizontal plane through the platform is $R(\theta) = (u^2/g)\sin 2\theta + (2uV/g)\sin\theta$, where $u = \sqrt{u_1^2 + u_2^2}$ and $\theta\,(0 < \theta < \pi/2)$ is the angle of projection. Also determine the angle θ for which the range is maximum.

(b) A particle of mass m is moving in an isotropic central force field $F(r)\,\hat{\mathbf{r}}$.

i. Show that the orbit would be a circle of radius r_0 if $F(r_0) = -L^2/mr_0^3$, where L is the angular momentum of the particle.

ii. Verify that the orbit is stable if $F'(r_0) < -3F(r_0)/r_0$.

[7+(4+3)]

3. (a) Show that $J_{\pm\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}}\cos\left(\dfrac{\pi}{4} - x \pm \dfrac{\pi}{4}\right)$, where $J_\nu(x)$ denotes Bessel function of order ν.

(b) Evaluate $\displaystyle\int_0^1 xP_3(x)\,dx$ where $P_3(x)$ denotes Legendre polynomial of order 3.

[7+7]

4. (a) Consider the differential equation

$$p_0(x)\frac{d^2u(x)}{dx^2} + p_1(x)\frac{du(x)}{dx} + p_2(x)u(x) = 0$$

where $p_i(x)$, $i = 0, 1, 2$ are real functions of x over $[a, b]$ and the first $(2 - i)$ derivatives of $p_i(x)$ are continuous.

i. Express the above equation in self adjoint form.

ii. Show that if $u_1(x)$ is a solution, then a second solution is given by $u_2(x) = C\, u_1(x) \int^x \frac{dt}{p_0(t)u_1^2(t)}$, where C is a constant.

(b) Solve the partial differential equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2xy$$

given that $u(x, y) = 2$ on $y = x^2$.

[(3+4)+7]

5. (a) Expand the function

$$f(x) = x, \quad 0 < x < 2$$

in a half range Sine series and hence obtain a Fourier series of the function $\phi(x) = x^2$ in the same interval.

(b) Let $f(x, y)$ be a real function of two variables defined as follows

$$f(x,y) = \begin{cases} \dfrac{3xy^2}{x^3 + y^3} & \text{if } (x, y) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Is f continuous? Justify your answer.

[7+7]

6. (a) Let G be a commutative group of order 6. Show that G is cyclic.

(b) An integer $x \in \mathbf{Z}_n - \{0\}$ is said to be a *quadratic residue* modulo n if there is $y \in \mathbf{Z}_n$ such that $y^2 \equiv x \bmod n$. Find all the quadratic residues modulo 17.

[8+6]

7. (a) Let $f : [0,1] \to [0,1]$ be a function from the unit interval into itself with the property that

$$|f(x) - f(y)| \leq \lambda |x - y|, \forall\, x, y \in [0,1], \quad \text{when } \lambda < 1.$$

Show that f is continuous on $[0,1]$ and hence has a fixed point. Is the fixed point unique? Justify your answer.

(b) Solve the following system of equations

$$7x + 4y = 8 \ ; \ 3x + 5y = 2$$

over $\mathbf{Z}_{13}$, the field of integers modulo 13.

[8+6]

8. (a) Find the eigenvalues of the following matrix.

$$A = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

(b) Find all real numbers α such that $(\alpha I + A)$ is invertible, where I is the 4×4 identity matrix.

(c) Fix an α such that $B = (\alpha I + A)$ is invertible. Find its inverse.
(Hint: Assume that the inverse is of the same form as B.)

[6+3+5]

Part III

Physics

Answer any five questions.

1. (a) A particle of mass m moves in a central potential $V(r) = \kappa r^n$, where r is the distance of the particle from the center and κ and n are constants. The particle has a constant angular momentum J. Show that for a circular orbit the radius r_0 satisfies the relation $(r_0)^{n+2} = J^2/(mn\kappa)$.

   (b) Synchronized clocks A and B are at rest in an inertial reference frame. Clock C is moving with velocity $(3/5)c$ along the line joining A and B, c being the velocity of light in vacuum. When C passes A, both the clocks A and C read $t = 0$. Answer the following questions.

      i. What time does C read when it reaches B?

      ii. How far apart are A and B in the inertial frame in which clock C is at rest?

      iii. In C's frame, when A passes C, what time does B read?

[8+(2+2+2)]

2. (a) A rocket with mass M_0 and loaded with fuel of mass m_0 takes off vertically upwards in a uniform downward gravitational field. The rocket ejects fuel downwards at velocity U_0 with respect to itself. The fuel is completely ejected in time t_0. Find the velocity of the rocket at time t_0 in terms of the above parameters and g, acceleration due to gravity.

   (b) Consider two pendulums hanging parallelly from a rigid support, with the bobs connected by a spring with spring constant k. The whole system moves in a plane. Assuming the bobs to have equal mass and each string of length l, show that the system can have at least one mode of frequency matching the normal frequency of each individual pendulum $\sqrt{g/l}$.

[6+8]

3. (a) Two positive charges q_1 and q_2 are placed at a distance r apart. A third charge q is placed such that all the three charges are in equilibrium. Find the magnitude and position of charge q.

(b) Three concentric spherical metallic shells A, B and C of radii a, b and c ($a < b < c$) have surface charge densities σ, $-\sigma$ and σ, respectively. Find the potentials at any point on the circumference for three individual spherical shells.

(c) Consider a gas of free, non-interacting electrons (each of mass m and charge e) of density n. The gas is in the presence of an electric field $\vec{E}e^{-i\omega t}$ with constant ω. Determine the current induced by the electric field.

[4+5+5]

4. (a) A system of two energy levels E_0 and E_1 ($E_1 > E_0 > 0$) is populated by N particles at temperature T. The particles populate the energy levels according to the classical distribution law.

   i. Derive an expression for the average energy per particle.

   ii. Derive an expression for the specific heat of the system of N particles.

   (b) One mole of a monatomic perfect gas initially at temperature T_0 expands (i) at constant temperature and (ii) at constant pressure from volume V_0 to $2V_0$. Calculate the work of expansion and the heat absorbed by the gas in each case.

[(3+4)+(3+4)]

5. (a) A material is brought from temperature T_i to temperature T_f by placing it in contact with a series of N reservoirs at temperature $T_i + \delta T$, $T_i + 2\delta T$, $\ldots$ $T_i + N\delta T = T_f$. Assuming that the heat capacity of the material C is temperature independent, calculate the entropy change of the total system, material plus reservoirs. What is the entropy change for $N \to \infty$ for fixed $T_f - T_i$?

   (b) Derive an expression for the chemical potential of a free electron gas in 3D with a density of N electrons per unit volume at zero temperature ($T = 0\,\mathrm{K}$). Find the chemical potential, in unit of electron volt, of the conduction electrons (which can be considered as free electrons) in a metal with $N = 10^{22}$ electrons/cm^3 at $T = 0\,\mathrm{K}$. Mass of an electron $= 9.11 \times 10^{-31}$ Kg and Planck's constant $= 6.53 \times 10^{-34}$ J-s.

[6+(4+4)]

6. (a) Two spin-$\frac{1}{2}$ particles A and B form a composite system. A is in the eigenstate $S_z = +1/2$ and B in the eigenstate $S_x = +1/2$. What is the probability that the total spin of the system will give the value zero?

   (b) Consider a particle with angular momentum j. Then for any simultaneous eigen state of the operators of J^2 and J_z, show that the expectation value satisfies the following relation

$$\langle J_x \rangle = \langle J_y \rangle = 0.$$

   (c) An anharmonic one-dimensional quantum oscillator for a particle of mass m has potential $V(x) = \frac{1}{2}m\omega^2 x^2 + \lambda x^4$, where $\lambda > 0$ is small. Using perturbation theory, determine the ground state energy to first order in λ.

[6+4+4]

7. (a) Consider a particle of mass m subject to the potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ \infty & \text{for } x > L \\ 0 & \text{for } 0 \leq x \leq L. \end{cases}$$

   The wave function of the particle at $t = 0$ is given by

$$\psi(x) = \frac{2}{\sqrt{L}} \cos\left(\frac{3\pi x}{L}\right) \sin\left(\frac{4\pi x}{L}\right).$$

   i. Find the expectation value for energy.
   ii. Find the wave function at $t = T$.

   (b) Consider the following wave function of the simple harmonic oscillator

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\mu|^2} \frac{\mu^n}{\sqrt{n!}} |n\rangle$$

   where $|n\rangle$ is an eigen function of the Hamiltonian with eigenvalue $\left(n + \frac{1}{2}\right)\hbar\omega$ and μ is a complex number. Show that the expectation value of the number operator is given by

$$\langle\phi|\hat{N}|\phi\rangle = |\mu|^2.$$

[(4+4)+6]

8. (a) Consider the Lagrangian of a charged scalar field $\phi(x)$ with an interaction term $\lambda(\phi^*\phi)^3$. Here λ and m are the coupling constant and mass parameter, respectively.

   i. Write down the equations of motion.

   ii. Derive an expression for the charge current and show that it is conserved.

   (b) Which of the following reactions violate a conservation law? Where there is a violation, mention the law that is violated.

   i. $\pi^0 \rightarrow \gamma + \gamma + \gamma$

   ii. $\pi^+ \rightarrow \mu^+ + \nu_\mu$

   iii. $\pi^+ \rightarrow \mu^+ + \bar{\nu}_\mu$

[(3+5)+(2+2+2)]