2014

BOOKLET No. TEST CODE : MTB

Afternoon

Answer as many questions as you can

Answering *five* questions correctly would be considered adequate

Time : 2 hours

Write your Registration number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answer-booklet.

ALL ROUGH WORK IS TO BE DONE ON THIS BOOKLET
AND/OR THE ANSWER-BOOKLET.
CALCULATORS ARE NOT ALLOWED.

STOP! WAIT FOR THE SIGNAL TO START.

- $\mathbb{Q}$ denotes the set of rational numbers.

Q 1. Let X, Y be normed linear spaces. Let $T : X \to Y$ be a linear map. Suppose T is an open mapping. Is T necessarily onto? Justify your answer.

Q 2. Let $M : L^2[0,1] \to L^2[0,1]$ be the linear operator defined by

$$(Mf)(x) = xf(x),$$

for $f \in L^2[0,1]$ and $x \in [0,1]$. Prove that M is bounded but not compact.

Q 3. Let A be an orthogonal matrix partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is a square matrix. Prove that A_{11} is invertible if and only if A_{22} is invertible.

Q. 4 Let A be a $n \times n$ nonsingular matrix with all eigenvalues real and

$$\text{Trace}(A^2) = \text{Trace}(A^3) = \text{Trace}(A^4).$$

Find Trace(A).

Q. 5 Let p be a prime number. Given any positive integer n, write its p-ary expansion as

$$n = \sum_{i=0}^{n} a_i p^i,$$

where $0 \leq a_i < p$. Show that the largest number e such that p^e divides $n!$ is given by the formula

$$e = \sum_{i=1}^{n} a_i \frac{p^i - 1}{p - 1}.$$

Q 6. Let S_n be the permutation group on $\{1, \ldots, n\}$. For $\pi \in S_n$, define

$$I(\pi) = \{\{i, j\} \subset \{1, \ldots, n\} : \frac{\pi(j) - \pi(i)}{j - i} < 0\},$$

and $i(\pi) = \#(I(\pi))$, the number of elements in $I(\pi)$. Prove that

$$I(\pi \circ \sigma) = I(\sigma) \triangle \sigma^{-1}(I(\pi)),$$

for each $\pi, \sigma \in S_n$. Hence prove that sgn : $S_n \to \{-1, 1\}$, defined by

$$\mathrm{sgn}(\pi) = (-1)^{i(\pi)},$$

is a group homomorphism.

[For any two sets A and B, the symmetric difference set $A \triangle B$ is defined as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
Also in the above $\sigma^{-1}(I(\pi)) = \{\{\sigma^{-1}(i), \sigma^{-1}(j)\} : \{i, j\} \in I(\pi)\}$.]

Q 7. Let A be a finite set. For $0 \leq i \leq 2$, let a_i be the number of subsets B of A such that

$$\#(B) \equiv i \pmod 3.$$

Prove that

$$|a_i - a_j| \leq 1,$$

for all $0 \leq i, j \leq 2$.

Q 8. Let $f(x)$ be an irreducible polynomial of degree 6 over a field K. Let L be an extension field of K of degree 2. Prove that f is either irreducible over L, or f factors into two irreducible cubic polynomials over L.

Q 9. A subgroup M of a group G is said to be maximal if $M \neq G$ and whenever H is a subgroup of G with $M \subseteq H \subseteq G$, then either $H = M$ or $H = G$. Prove that the additive group $(\mathbb{Q}, +)$ has no maximal subgroup.

Q 10. Suppose $f \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$. Show that there is no complex number α such that $f(\alpha) = f(\alpha + 1) = 0$.